Warm-Up

CST Algebra I 11.0

Which is a factor of $2x^2 - x - 21$?

A. $(x+7)$
B. $(x+3)$
C. $(x-7)$
D. $(x-3)$

Find the other factor for $2x^2 - x - 21$.

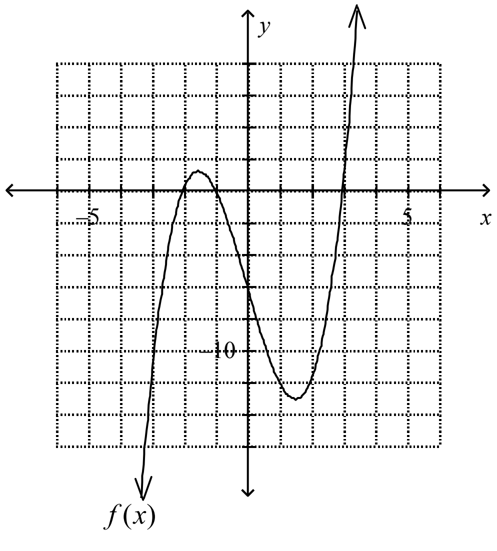
Algebra II 3.0

Divide two different ways:

$$\frac{3x^2 - 5x - 2}{x - 2}$$

Algebra II 10.0

Which values of x appear to be zeros of $f(x)$?



Based on your answer, state three factors of $f(x)$.

Algebra II 8.0

Find the roots for the polynomial below:

$$x^2 - 3x + 1$$

Are the roots are rational or irrational?

Warm Up Solutions

CST Algebra I 11.0

$$\begin{aligned} &2x^2-x-21 \\ &=(2x-7)(x+3) \end{aligned}$$

Answer: B

The other factor is $(2x-7)$

Algebra II 3.0

$$\frac{3x^2-5x-2}{x-2}$$

$$\begin{aligned} &\frac{3x^2-5x-2}{x-2} \\ &=\frac{(3x+1)(x-2)}{x-2} \\ &=3x+1 \end{aligned}$$

$$\begin{array}{r} 3x+1 \\ x-2\overline{\big)3x^2-5x-2} \\ \underline{-\left(3x^2-6x\right)} \\ x-2 \\ \underline{-\left(x-2\right)} \\ 0 \end{array}$$

Algebra II 10.0

The zeros of $f(x)$ appear to be -2, -1, and 3.

The factors are $\left(x+2\right)$, $\left(x+1\right)$, and $\left(x-3\right)$

Algebra II 8.0, 10.0

The roots are found by setting the polynomial equal to zero, then solving.

$$x^2-3x+1=0$$

Using the quadratic formula:

$$a=1,\ b=-3,\ c=1$$

$$\begin{aligned} x&=\frac{3\pm\sqrt{(-3)^2-4(1)(1)}}{2(1)} \\ &=\frac{3\pm\sqrt{9-4}}{2} \\ &=\frac{3\pm\sqrt{5}}{2} \end{aligned}$$

The roots are irrational.

Polynomial Division

Long Division

We can divide polynomials using the long division algorithm, in a similar fashion to using long division for numerical expressions.

Example 1)

Numeric Long Division	**Polynomial Long Division**
$4527 \div 13$	$\frac{6x^2-26x+12}{x-4}$
$\begin{array}{r} 348 \\ 13\overline{)4527} \\ -\ 39 \\ \hline 62 \\ -\ \ 52 \\ \hline 107 \\ -\ 104 \\ \hline 3 \end{array}$	$\begin{array}{r} 6x\ -2 \\ x-4\overline{)6x^2-26x+12} \\ -(6x^2-24x) \\ \hline -2x+12 \\ -\ (-2x+\ 8) \\ \hline 4 \end{array}$
$4527 \div 13 = 348+\frac{3}{13}$ $\therefore\ 4527 = 13(348)+3$	$\frac{6x^2-26x+12}{x-4} = 6x-2+\frac{4}{x-4}$ $\therefore\ 6x^2-26x+12=(x-4)(6x-2)+4$

Example 2) Divide using long division.

$$\frac{2x^3+7x^2-5}{x+3}$$

Notice that the linear term in the dividend is missing. Rewrite the numerator, inserting 0*x*. This allows like terms to line up during the long division process.

$$\frac{2x^3+7x^2+0x-5}{x+3}$$

$$\begin{array}{r} 2x^2+\ x-3 \\ x+3\overline{)2x^3+7x^2+0x-5} \\ -(2x^3+6x^2) \\ \hline x^2+0x \\ -\ (x^2+3x) \\ \hline -3x-5 \\ -(-3x-9) \\ \hline 4 \end{array}$$

$$\frac{2x^3+7x^2-5}{x+3} = 2x^2+x-3+\frac{4}{x+3}$$

$$\therefore\ 2x^3+7x^2-5=(x+3)(2x^2+x-3)+4$$

Example 3) **You try:** Divide using long division.

$$\frac{4x^3-9x+5}{x-2}$$

$$\begin{array}{r} 4x^2+8x\ +7 \\ x-2\overline{)4x^3+0x^2-9x\ +5} \\ \underline{-(4x^3-8x^2)} \\ 8x^2-9x \\ \underline{-(8x^2-16x)} \\ 7x+5 \\ \underline{-(7x-14)} \\ 19 \end{array}$$

$$\frac{4x^3-9x+5}{x-2}=4x^2+8x\ +7+\frac{19}{x-2}$$

$$\therefore\ 4x^3-9x+5=(x-2)\left(4x^2+8x\ +7\right)+19$$

Synthetic Division

Synthetic division can be used when the divisor is in the form $x-a$. Notice in long division that we are only manipulating the coefficients; the variables are "holding the place."

Process: For a divisor in the form $x-a$, start with $-a$. Then write the coefficients of the dividend, remembering to write 0 for any missing terms.

We will do example 2 again, using synthetic division:

$$\frac{2x^3+7x^2-5}{x+3}$$

$$=\frac{2x^3+7x^2+0x-5}{x+3}$$

Long Division	**Synthetic Division**
$\begin{array}{r} 2x^2+x-3 \\ x+3\overline{)2x^3+7x^2+0x-5} \\ \underline{-(2x^3+6x^2)} \\ x^2+0x \\ \underline{-(x^2+3x)} \\ -3x-5 \\ \underline{-(-3x-9)} \\ 4 \end{array}$ $\therefore \frac{2x^3+7x^2-5}{x+3}=2x^2+x-3+\frac{4}{x+3}$	Since the divisor is $x+3$, $-a$ is -3 $\begin{array}{r\|rrrr} -3 & 2 & 7 & 0 & -5 \\ & & -6 & -3 & 9 \\ \hline & \underbrace{2 \quad 1 \quad -3}_{\text{Coefficients of the quotient}} & & & \downarrow \\ & & & & \text{remainder} \end{array}$ Use the opposite of *a* $\therefore \frac{2x^3+7x^2-5}{x+3}=2x^2+x-3+\frac{4}{x+3}$

Notice that the first term in the quotient is degree 2, which is one degree less than the dividend.

Example 4) **You try:** Divide using long division and synthetic division.

$$\frac{2x^3+x^2-13x+6}{x-2}$$

Long Division	**Synthetic Division**
$\begin{array}{r} 2x^2+5x-3 \\ x-2\overline{)2x^3+x^2-13x+6} \\ \underline{-(2x^3-4x^2)} \\ 5x^2-13x \\ \underline{-(5x^2-10x)} \\ -3x+6 \\ \underline{-(-3x+6)} \\ 0 \end{array}$ $\therefore \frac{2x^3+x^2-13x+6}{x-2}=2x^2+5x-3$	$\begin{array}{r\|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & \underbrace{2 \quad 5 \quad -3}_{\text{Coefficients of the quotient}} & & & \downarrow \\ & & & & \text{remainder} \end{array}$ the opposite of *a* $\therefore \frac{2x^3+x^2-13x+6}{x-2}=2x^2+5x-3$

Rational Root Theorem

Notice in example 3, the remainder is zero. This means we can factor the polynomial:

$$\frac{2x^3+x^2-13x+6}{x-2}=2x^2+5x-3$$

$$\therefore\ 2x^3+x^2-13x+6=(x-2)(2x^2+5x-3)$$

Since $2x^2+5x-3$ can also be factored,

$$\begin{aligned}2x^3+x^2-13x+6&=(x-2)(2x^2+5x-3)\\&=(x-2)(2x-1)(x+3)\end{aligned}$$

The Rational Root Theorem can be used to find the rational roots of a polynomial, and can therefore aid in factoring.

Each rational root can be written as a fraction $\frac{p}{q}$, where

- p is a factor of the constant term of the polynomial
- q is a factor of the leading coefficient.

To find a rational root, we can start with a list of possible rational roots of the equation. Choose one of the possible rational roots, then use synthetic division to determine if it is in fact a root.

The roots of a polynomial are also called the zeros of the related function.

Example 5) Factor $2x^4-3x^3-20x^2+27x+18$ and find the zeros for the related function.

By the Rational Root Theorem, the possible rational roots are:

$$\frac{\pm1,\pm2,\pm3,\pm6,\pm9,\pm18}{\pm1,\pm2}$$

$$=1,-1,2,-2,3,-3,6,-6,9,-9,18,-18,\frac{1}{2},-\frac{1}{2},\frac{3}{2},-\frac{3}{2},\frac{9}{2},-\frac{9}{2}$$

Determine if $x=1$ is a root by using synthetic division. If the remainder is zero, then 1 is a root.

$$\begin{array}{r|rrrrr}\underline{1} & 2 & -3 & -20 & 27 & 18\\ & & 2 & -1 & -21 & 6\\\hline & 2 & -1 & -21 & 6 & 24\end{array}$$

The remainder is not 0, so try another possible rational root.

Try $x=-1$:

$$\begin{array}{r|rrrrr} -1 & 2 & -3 & -20 & 27 & 18 \\ & & -2 & 5 & 15 & -42 \\ \hline & 2 & -5 & -15 & 42 & -24 \end{array}$$

The remainder is not 0, so try another possible rational root.

Try $x=2$:

$$\begin{array}{r|rrrrr} 2 & 2 & -3 & -20 & 27 & 18 \\ & & 4 & 2 & -36 & -18 \\ \hline & 2 & 1 & -18 & -9 & 0 \end{array}$$

Since the remainder is zero, $x=2$ is a root for the polynomial, and $(x-2)$ is a factor.

$$2x^4-3x^3-20x^2+27x+18=(x-2)\left(2x^3+x^2-18x-9\right)$$

Continue to factor $2x^3+x^2-18x-9$

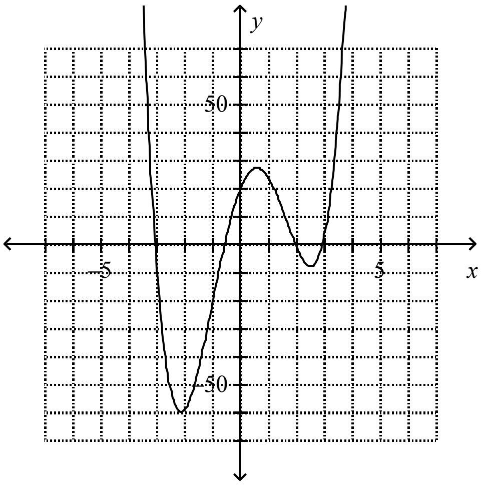
Try $x=3$:

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -18 & -9 \\ & & 6 & 21 & 9 \\ \hline & 2 & 7 & 3 & 0 \end{array}$$

Since the remainder is zero, $x=3$ is a root for the polynomial, and $(x-3)$ is a factor.

$$\begin{aligned} 2x^4-3x^3-20x^2+27x+18 &= (x-2)(x-3)\left(2x^2+7x+3\right) \\ &= (x-2)(x-3)(x+3)(2x+1) \end{aligned}$$

Also, the zeros for the related function $f(x)=2x^4-3x^3-20x^2+27x+18$ are $x=-3, x=-\frac{1}{2}, x=2, x=3$.



Example 6) **You try:** Factor $x^4+11x^3+41x^2+61x+30$ and find the zeros for the related function.

By the Rational Root Theorem, the possible rational roots are:

$$\frac{\pm1,\pm2,\pm3,\pm10,\pm15,\pm30}{\pm1}$$
$$=1,-1,2,-2,3,-3,5,-5,10,-10,15,-15,30,-30$$

Try $x=-1$:

$$\begin{array}{r|rrrrr} -1 & 1 & 11 & 41 & 61 & 30 \\ & & -1 & -10 & -31 & -30 \\ \hline & 1 & 10 & 31 & 30 & 0 \end{array}$$

$\therefore x=-1$ is a root for the polynomial, and $(x+1)$is a factor

$$x^4+11x^3+41x^2+61x+30=(x+1)(x^3+10x^2+31x+30)$$

Continue to factor:

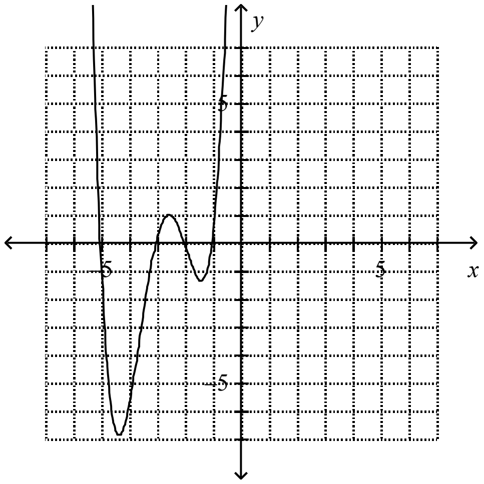
Try $x=-2$:

$$\begin{array}{r|rrrr} -2 & 1 & 10 & 31 & 30 \\ & & -2 & -16 & -30 \\ \hline & 1 & 8 & 15 & 0 \end{array}$$

$\therefore x=-2$ is a root for the polynomial, and $(x+2)$is a factor

$$\begin{aligned} x^4+11x^3+41x^2+61x+30 &= (x+1)(x+2)(x^2+8x+15) \\ &= (x+1)(x+2)(x+3)(x+5) \end{aligned}$$

The zeros for the related function $f(x)=x^4+11x^3+41x^2+61x+30$ are $x=-1, x=-2, x=-3, x=-5$



Other Methods for Polynomial Division

I. Generic Area Model (Grid)

Example 7) Divide $(2x^2+10x+15)\div(x+3)$

Set up:

Place the first term of the dividend in the upper left corner of the grid. Write the terms of the divisor if front of each row. Leave space along the top to write the quotient.

x	$2x^2$		
$+3$			

Process:

Divide the first term in the dividend by the first term in the divisor. This is the first term of the quotient. Record the result above the first column.

	$2x$		
x	$2x^2$		
$+3$			

Multiply the first term of the quotient by the second term of the divisor and record in the 1st position of the second row.

	$2x$		
x	$2x^2$		
$+3$	$6x$		

Determine what needs to be added to $6x$ so that the sum is equal to $10x$, the second term in the dividend. $4x+6x=10x$.

	$2x$		
x	$2x^2$	4x	
$+3$	$6x$		

The sum must be $10x$

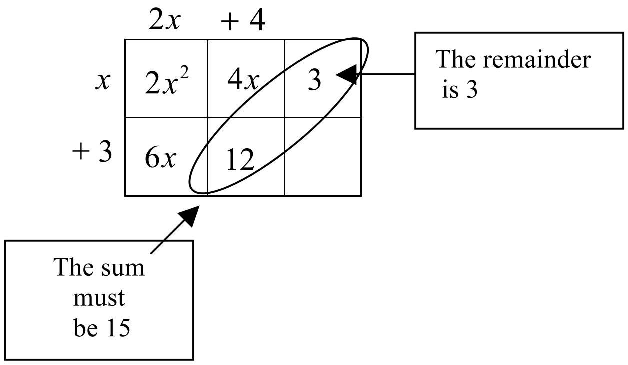
Repeat the process:
Divide the second term in the top row by the first term in the divisor. This is the second term of the quotient. Record the result above the second column.

	$2x$	$+ 4$	
x	$2x^2$	$4x$	
$+ 3$	$6x$		

Multiply the second term of the quotient by the second term of the divisor and record in the 2nd position of the second row.

	$2x$	$+ 4$	
x	$2x^2$	$4x$	
$+ 3$	$6x$	12	

Determine what needs to be added to 12 so that the sum is equal to 15, the third term in the dividend.
3 + 12 = 15.   This is the remainder.



	$2x$	$+ 4$	
x	$2x^2$	$4x$	3
$+ 3$	$6x$	12	

The remainder is 3

The sum must be 15

Answer: $(2x^2 + 10x + 15) \div (x + 3) = (2x + 4) + \dfrac{3}{x + 3}$

Example 8) Divide $(4x^3+10x^2-7)\div(2x-1)$

	$2x^2$			
$2x$	$4x^3$			
-1				

	$2x^2$			
$2x$	$4x^3$			
-1	$-2x^2$			

	$2x^2$			
$2x$	$4x^3$	$12x^2$		
-1	$-2x^2$			

Sum must be $10x^2$

	$2x^2$	$6x$		
$2x$	$4x^3$	$12x^2$		
-1	$-2x^2$			

	$2x^2$	$6x$		
$2x$	$4x^3$	$12x^2$	$6x$	
-1	$-2x^2$	$-6x$		

Sum must be $0x$

	$2x^2$	$6x$	3	
$2x$	$4x^3$	$12x^2$	$6x$	
-1	$-2x^2$	$-6x$		

	$2x^2$	$6x$	3	
$2x$	$4x^3$	$12x^2$	$6x$	
-1	$-2x^2$	$-6x$	-3	

	$2x^2$	$6x$	3	
$2x$	$4x^3$	$12x^2$	$6x$	-4
-1	$-2x^2$	$-6x$	-3	

Sum must be -7

Answer: $(4x^3+10x^2-7)\div(2x-1)=2x^2+6x+3+\dfrac{-4}{2x-1}$

Example 9) **You Try:** $(-3x^3+5x-8)\div(x-2)$

	$-3x^2$	$-6x$	-7	
x	$-3x^3$	$-6x^2$	$-7x$	-22
-2	$6x^2$	$12x$	14	

Answer: $(-3x^3+5x-8)\div(x-2)=-3x^2-6x-7-\dfrac{22}{x-2}$

II. Using Definition of Division:

$$\boxed{\text{Dividend} = (\text{Divisor})(\text{ Quotient}) + \text{Remainder}}$$

Example 10) Divide $(2x^2+10x+15)\div(x+3)$

$$\begin{aligned}(2x^2+10x+15) &= (x+3)(ax+b)+R\\ &= ax^2+bx+3ax+3b+R\\ &= ax^2+(b+3a)x+(3b+R)\end{aligned}$$

Since the dividend is $2x^2+10x+15$, we can find the values of *a*, *b*, and *R* as follows:

$$\begin{array}{ccccc} ax^2 & + & (b+3a)x & + & (3b+R)\\ \downarrow & & \downarrow & & \downarrow\\ 2x^2 & + & 10x & + & 15\end{array}$$

$$\begin{array}{lll} a=2 & b+3a=10 & 3b+R=15\\ & b+3(2)=10 & 3(4)+R=15\\ & b+6=10 & 12+R=15\\ & b=4 & R=3\end{array}$$

Answer: $(2x^2+10x+15)\div(x+3)=2x+4+\dfrac{3}{x+3}$

Example 11) Divide $(4x^3+10x^2-7)\div(2x-1)$

$$\begin{aligned}(4x^3+10x^2-7) &= (2x-1)(ax^2+bx+c)+R\\ &= 2ax^3+2bx^2+2cx-ax^2-bx-c+R\\ &= 2ax^3+(2b-a)x^2+(2c-b)x+(-c+R)\end{aligned}$$

$$\begin{array}{llll} 2a=4 & 2b-a=10 & 2c-6=0 & -c+R=-7\\ a=2 & 2b-2=10 & 2c=6 & -3+R=-7\\ & 2b=12 & c=3 & R=-4\\ & b=6 & & \end{array}$$

$$(4x^3+10x^2-7)=(2x-1)(2x^2+6x+3)+-4$$

Answer: $(4x^3+10x^2-7)\div(2x-1)=2x^2+6x+3-\dfrac{4}{2x-1}$

You Try

Example 12) $(-3x^3+8x^2-7x+7)\div(3x-2)$

$$\begin{aligned}-3x^3+8x^2-7x+7&=(3x-2)(ax^2+bx+c)+R\\&=3ax^3+3bx^2+3cx-2ax^2-2bx-2c+R\\&=3ax^3+(3b-2a)x^2+(3c-2b)x+(-2c+R)\end{aligned}$$

$$\begin{array}{rrrr}3a=-3 & 3b-2a=8 & 3c-2b=-7 & -2c+R=7\\ a=-1 & 3b-2(-1)=8 & 3c-2(2)=-7 & -2(-1)+R=7\\ & 3b+2=8 & 3c-4=-7 & 2+R=7\\ & 3b=6 & 3c=-3 & R=5\\ & b=2 & c=-1 & \end{array}$$

$$-3x^3+8x^2-7x+7=(3x-2)(-x^2+2x-1)+5$$

Answer: $(-3x^3+8x^2-7x+7)\div(3x-2)=-x^2+2x-1+\dfrac{5}{3x-2}$

Example 13) Divide using several methods.

$$\frac{2x^4-9x^2+3x+5}{x-2}$$

Long Division

$$\begin{array}{r l}
 & 2x^3+4x^2\ -x\ \ +1 \\
x-2 \,\big)\! & 2x^4+0x^3-9x^2+3x+5 \\
 & \underline{-(2x^4-4x^3)} \\
 & 4x^3-9x^2 \\
 & \underline{-\ (4x^3-8x^2)} \\
 & -x^2\ +3x \\
 & \underline{-(-x^2\ +2x)} \\
 & x+5 \\
 & \underline{-\ (x-2)} \\
 & 7
\end{array}$$

$$\therefore \frac{2x^4-9x^2+3x+5}{x-2}=2x^3+4x^2\ -x\ \ +1+\frac{7}{x-2}$$

Synthetic Division

Since the divisor is $x-2$, $-a$ is 2

$$\begin{array}{c|ccccc}
2 & 2 & 0 & -9 & 3 & 5 \\
 & & 4 & 8 & -2 & 2 \\
\hline
 & 2 & 4 & -1 & 1 & 7
\end{array}$$

$$\therefore \frac{2x^4-9x^2+3x+5}{x-2}=2x^3+4x^2\ -x\ \ +1+\frac{7}{x-2}$$

Grid

	$2x^3$	$4x^2$	$-1x$	1	
x	$2x^4$	$4x^3$	$-x^2$	x	7
-2	$-4x^3$	$-8x^2$	$2x$	-2	

$$\therefore \frac{2x^4-9x^2+3x+5}{x-2}=2x^3+4x^2\ -x\ \ +1+\frac{7}{x-2}$$

Definition of Division

$$\begin{aligned}
&2x^4-9x^2+3x+5 \\
&=2x^4+0x-9x^2+3x+5 \\
&=(x-2)(ax^3+bx^2+cx+d)+R \\
&=ax^4+bx^3+cx^2+dx-2ax^3-2bx^2-2cx-2d+R \\
&=ax^4+(b-2a)x^3+(c-2b)x^2+(d-2c)x+(-2d+R)
\end{aligned}$$

$$\begin{array}{rrrrr}
a=2 & b-2a=0 & c-2b=-9 & d-2c=3 & -2d+R=5 \\
 & b-2(2)=0 & c-2(4)=-9 & d-2(-1)=3 & -2(1)+R=5 \\
 & b-4=0 & c-8=-9 & d+2=3 & -2+R=5 \\
 & b=4 & c=-1 & d=1 & R=7
\end{array}$$

$$\therefore \frac{2x^4-9x^2+3x+5}{x-2}=2x^3+4x^2\ -x\ \ +1+\frac{7}{x-2}$$